Exam – Finite Element Methods and Applications, January 28th 2025

The exam is composed by three exercises. Please justify all your answers. The final grade will be calculated by: points / 23 × 9 + 1. You have 2h to hand in your answers.

Exercise 1 (9 points)

Consider the following partial differential equation: Find $q(\cdot,t) : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, $d = \{2,3\}$, such that

$$\frac{\partial q}{\partial t} + q^3 + \alpha q - \nabla \cdot (\beta \nabla q) = f$$

with $\alpha(\cdot,t), \beta(\cdot,t), f(\cdot,t) : \Omega \to \mathbb{R}$ given functions, and $\alpha > 0$, $\beta > 0$.

(a) $\boxed{1.5}$ Give the weak form (continuous, not discrete) using the following boundary conditions:

$$q = a \ \text{ on } \Gamma_1 \ , \ \frac{\partial q}{\partial \mathbf{n}} = b \ \text{ on } \Gamma_2$$

with $\mathbf{n}$ the outwards normal vector to the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$.

(b) $\boxed{1.5}$ Assume now that the weak form you derived does not depend on time, namely that the time derivatives are zero. Show that bilinear form is $H^1(\Omega)$-elliptic.

(c) $\boxed{1.5}$ Compute the weak form (continuous, not discrete) using the following boundary conditions:

$$\frac{\partial q}{\partial \mathbf{n}} = \gamma q + c \ \text{ on } \partial\Omega$$

with $\gamma(\cdot,t) : \partial\Omega \to \mathbb{R}^- \backslash \{0\}$.

(d) $\boxed{1.5}$ For the weak problem derived in (c), prove that:

$$\int_\Omega |q(x,t)|^2 dx \to 0 \text{ when } t \to \infty.$$

when $f, c = 0$.

(e) $\boxed{2.0}$ For the weak problem derived in (c), assume now a time-discretization on the time points $t_1, t_2, \ldots$, with $\tau = t_k - t_{k-1}$. Propose a time-discretization method such that

$$\int_\Omega |q_k(x)|^2 dx \to 0 \text{ when } t_k \to \infty.$$

for $f, c = 0$, with $q_k(x)$ the approximation of $q(x, t_k)$. The resulting method should also be non-linear in $q_k(x)$, as in the continuous equation, at every t_k.

(f) $\boxed{1.0}$ By changing only the time-discretization of the non-linear term, propose another unconditionally stable time-discretization scheme leading to a linear problem for $q_k(x)$ at every t_k.

Exercise 2 (9 points)

The following velocity and pressure plots were generated from a finite element simulation of a transient Navier-Stokes problem, using $\mathbb{P}_1/\mathbb{P}_1$-PSPG-stabilized finite elements for velocity/pressure. The problem was discretized in time with a backward Euler-method (timestep $\tau = 0.001$) with semi-implicit treatment of the convective velocity. The meshes are shown in the figures. The weak formulation reads:

Given $\mathbf{u}^0 = \mathbf{0}$, find $\mathbf{u}^{k+1}(\cdot,t) : \Omega \to \mathbb{R}^2, p^{k+1}(\cdot,t) : \Omega \to \mathbb{R}$ for all $\mathbf{v}, q$, such that,

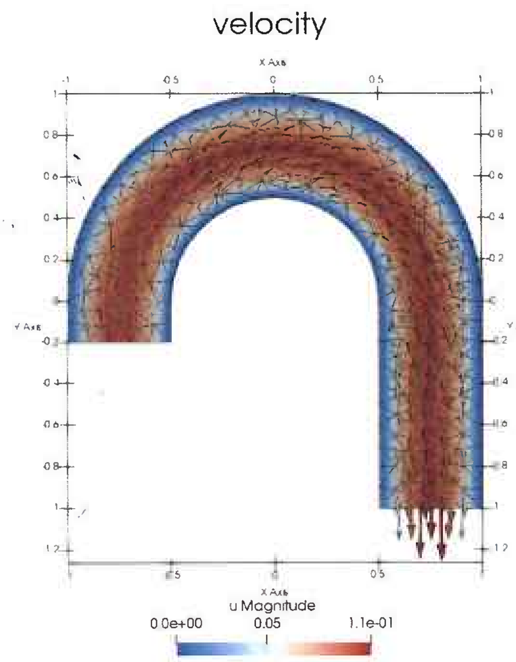
$$\frac{\rho}{\tau}\int_\Omega (\mathbf{u}^{k+1} - \mathbf{u}^k)\cdot\mathbf{v} + \rho\int_\Omega (\mathbf{u}^k\cdot\nabla)\mathbf{u}^{k+1}\cdot\mathbf{v} + \frac{\rho}{2}\int_\Omega (\nabla\cdot\mathbf{u}^k)\mathbf{u}^{k+1}\cdot\mathbf{v} + \int_\Omega \mu\nabla\mathbf{u}^{k+1} : \nabla\mathbf{v}$$
$$- \int_\Omega p^{k+1}\nabla\cdot\mathbf{v} + \int_\Omega \nabla\cdot\mathbf{u}^{k+1} q + \varepsilon\frac{h^2}{\mu}\int_\Omega \nabla p^{k+1}\cdot\nabla q = 0$$

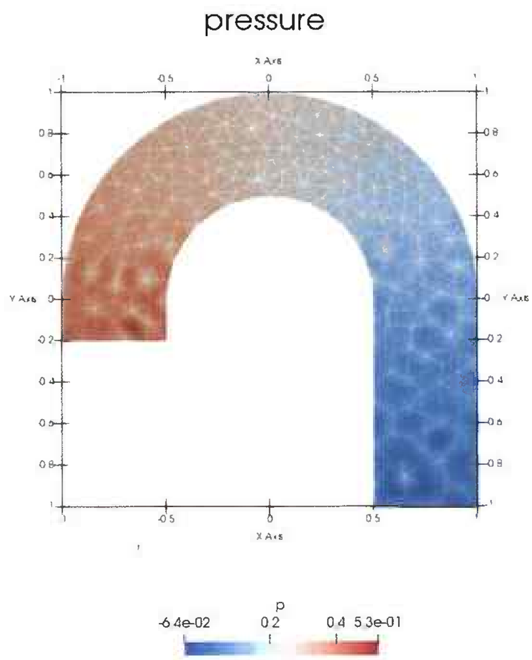
with boundary conditions:

$$\begin{aligned} \mathbf{u}^{k+1} &= \mathbf{g}(x) && \text{at the inlet (left horizontal line)}\\ \mathbf{u}^{k+1} &= \mathbf{0} && \text{at the walls (curved lines)}\\ \mu\mathbf{n}\cdot\nabla\mathbf{u}^{k+1} - p^{k+1}\mathbf{n} &= \mathbf{0} && \text{at the outlet (right horizontal line)} \end{aligned}$$

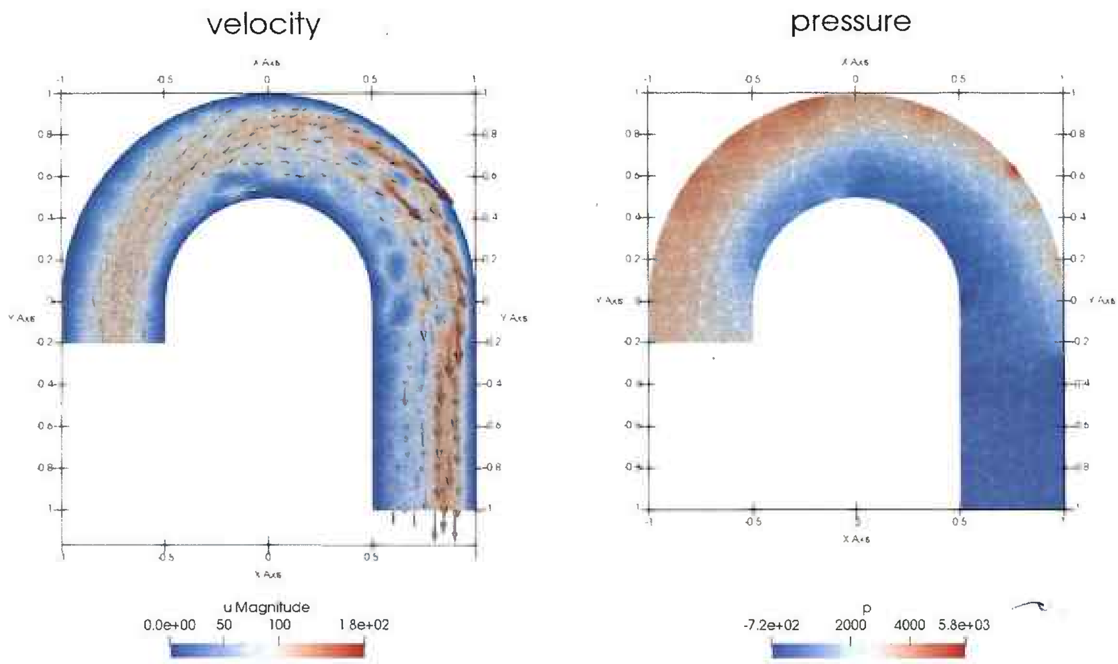
and with $\mathbf{v} = \mathbf{0}$ on the inlet and walls. The inlet velocity is constant in time, and the solution shown in the pictures is the steady state ($t \to \infty$). ε was chosen in the interval $[10^{-5}, 10^{-3}]$.

(a) [3] The following plots show the steady state solution ($t \to \infty$) computed for a Reynolds number of $Re = 1$. Indicate three strategies of modifying the discrete problem in order to reduce the spatial oscillations in the pressure. Give at least one advantage and one disadvantage for each of the strategies.



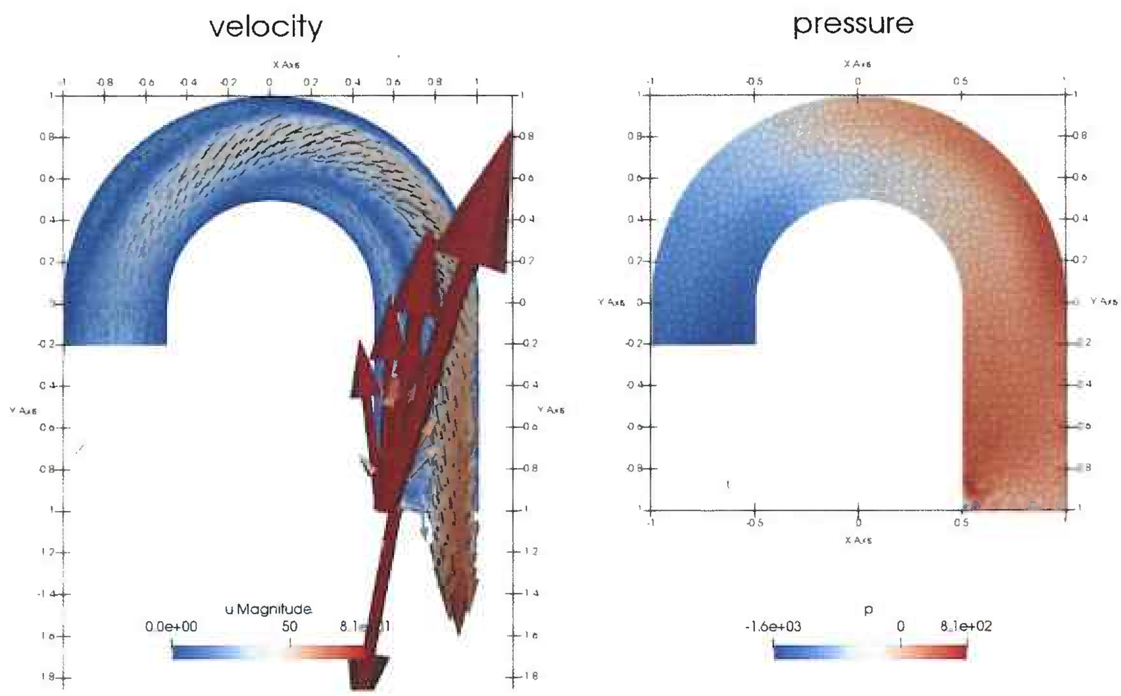


(b) [3] The following plots show the steady state solution ($t \to \infty$) computed for a Reynolds number of $Re = 1000$. Explain where the spatial oscillations in the velocity field come from. Indicate two strategies of modifying the discrete problem in order to reduce the spatial oscillations in the velocity. Explain what each of both strategies do from the perspective of the theoretical error bounds, for instance the best approximation. Give at least one advantage and one disadvantage for each of the strategies.



(c) $\boxed{3}$ The following plots show the solution at one time step of the solution computed for a pulsatile boundary condition at the inlet $\mathbf{u} = \sin(\omega t)\mathbf{g}(x)$. The solution is shown at the time close before $\omega t = \pi$, i.e., after the maximum velocity.

Explain the reason for the velocity instabilities, building up at the outlet. Indicate one strategy how to modify the original problem to eliminate the instabilities in the velocity. Give at least one advantage and one disadvantage of this strategy.



Exercise 3 (5 points)

Some techniques of medical imaging can measure the 3D velocity field of the blood flow over time, up to some measurement errors. We will aim to develop a mathematical and numerical model that allows to quantify these measurement errors by computing to what extent the measurements satisfy the incompressible Navier-Stokes equations (cool ah).

For this purpose, we will assume that the true flow velocity, denoted by $\vec{u}$, satisfies an incompressible Navier-Stokes problem in vessel lumen Ω:

$$\rho\frac{\partial\vec{u}}{\partial t} + \rho(\vec{u}\cdot\nabla)\vec{u} - \mu\Delta\vec{u} + \nabla p = 0 \quad \text{in} \quad \Omega \tag{1}$$

$$\nabla\cdot\vec{u} = 0 \quad \text{in} \quad \Omega \tag{2}$$

Another assumption is that, the velocity field measured denoted by $\vec{u}_{meas}$, satisfies the relation $\vec{u} = \vec{u}_{meas} + \vec{w}$, with $\vec{w}$ the measurement error and $\nabla\cdot\vec{w} = 0$. ρ and μ are supposed to be known.

Inserting $\vec{u}_{meas} + \vec{w}$ into equations (1)–(2) leads to: Find $\vec{w}$ and p such that

$$\rho\frac{\partial\vec{w}}{\partial t} + \rho((\vec{u}_{meas}+\vec{w})\cdot\nabla)\vec{w} + \rho(\vec{w}\cdot\nabla)\vec{u}_{meas} - \mu\Delta\vec{w} + \nabla p = -\rho\frac{\partial\vec{u}_{meas}}{\partial t} - \rho(\vec{u}_{meas}\cdot\nabla)\vec{u}_{meas} + \mu\Delta\vec{u}_{meas} \quad \text{in} \quad \Omega \tag{3}$$

$$\nabla\cdot\vec{w} = 0 \quad \text{in} \quad \Omega \tag{4}$$

We will assume that the boundary condition for $\vec{w}$ is $\vec{w} = \vec{0}$ on $\partial\Omega$.

(a) [3] Obtain the weak form of the set equations (3)-(4). Write it in such a way that all second derivatives in space are reduced to first order derivatives. Include the boundary condition.

(b) [2] For the weak problem you obtained in (a), assume zero right-hand-side, and denote this problem (PW). Starting from (PW), compute an expression for

$$\int_\Omega \|\vec{w}(x,t)\|^2 dx. \tag{5}$$

Assume also that the mass conservation for $\vec{w}$ in the weak form is satisfied at every spatial point, i.e. $\nabla\cdot\vec{w} = 0$. Indicate which terms of expression (5) may make it increase over time.